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HEAT CONDUCTION AND HEAT TRANSFER IN TECHNOLOGICAL PROCESSES

ANALYTICAL SOLUTION METHOD FOR HEAT CONDUCTION PROBLEMS BASED ON THE INTRODUCTION OF THE TEMPERATURE PERTURBATION FRONT AND ADDITIONAL BOUNDARY CONDITIONS

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With the use of the integral heat balance method based on the introduction of the temperature perturbation field and additional boundary conditions, we consider a method for finding analytical solutions of boundary-value problems of nonstationary heat conduction that permits obtaining, for a series of problems, solutions with a given degree of accuracy throughout the range of variation of the Fourier number. Solutions have a simple form of exponential algebraic polynomials, which makes it possible to investigate the heat transfer in the fields of isothermal lines, as well as analyze the time distributions of velocities of motion of isotherms.

Keywords: *analytical methods, temperature perturbation front, additional boundary conditions, integral methods, isotherms, velocities of travel of isotherms.*

It is known that solutions of heat conduction problems obtained by classical methods are given in the form of infinite series poorly converging in the vicinity of boundary points and at small values of the time coordinate. Investigations show that convergence of the exact analytical solution of a nonstationary heat conduction problem for an infinite plate at boundary conditions of the first kind in the range of Fourier numbers $10^{-12} \le \text{Fo} \le 10^{-7}$ is only observed when from 1000 ($\text{Fo} = 10^{-7}$) to 500,000 ($\text{Fo} = 10^{-12}$) series terms are used.

This problem is even more characteristic of variation methods (Ritz, Trefftz, L. V. Kantorovich, etc.) as well as of weighted residuals methods (Bubnov–Galerkin orthogonal method, moments method, collocation method, etc.). These methods are practically inapplicable for obtaining solutions of nonstationary heat conduction problems at small values of the time coordinate because, at a large number of approximations of the relatively unknown coefficients of the sought solution, systems of algebraic linear equations, whose number corresponds to the number of approximations, are obtained. Matrices of coefficients of such systems, being filled with square matrices with a significant spread of coefficients in absolute value, are ill-posed, as a rule. Therefore, with increasing number of approximations the accuracy of the solution may even worsen [1, 2].

Among the methods that permit avoiding the above difficulties are the integral heat balance methods [3–9]. However, their wide application is inhibited by the insufficient accuracy of solutions obtained, and all attempts to increase it did not lead to desirable results [3, 4].

Below we describe a method, belonging to the group of integral heat balance methods, that permits obtaining analytical solutions of boundary-value problems practically with a given degree of accuracy throughout the time interval of the nonstationary process $0 \le \text{Fo} < \infty$ without any restrictions on the value of the Fourier number in the range of its small values [8, 9]. Let us consider the main idea of the method with the example of solving the heat conduction problem for an infinite plate in the following mathematical formulation:

$$\text{Í}\frac{\partial \Theta(\xi, \text{Fo})}{\partial \text{Fo}} = \frac{\partial^2 \Theta(\xi, \text{Fo})}{\partial \xi^2} \quad (\text{Fo} > 0\,, \quad 0 < \xi < 1)\,, \tag{1}$$

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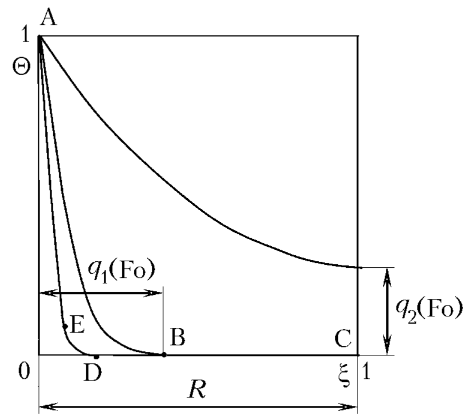

Fig. 1. Computational scheme of the heat exchange.

$$\Theta (\xi, 0) = 0 , \tag{2}$$

$$\Theta (0, \mathrm{Fo}) = 1 , \tag{3}$$

$$\frac{\partial \Theta (1, \mathrm{Fo})}{\partial \xi} = 0 . \tag{4}$$

Let us divide the heating process into two time stages: $0 < \mathrm{Fo} \le \mathrm{Fo}_1$ and $\mathrm{Fo}_1 \le \mathrm{Fo} < \infty$. To this end, let us introduce a moving with time boundary (temperature perturbation front) dividing the initial region $0 \le \xi \le 1$ into two subregions $0 \le \xi \le q_1(\mathrm{Fo})$ and $q_1(\mathrm{Fo}) \le \xi \le 1$, where $q_1(\mathrm{Fo})$ is a function defining the advance with time of the interface (Fig. 1). At the same time in the region situated behind the temperature perturbation front the initial temperature is preserved. The first stage of the process ends as soon as the moving boundary reaches the center of the plate ($\xi = 1$), i.e., when $\mathrm{Fo} = \mathrm{Fo}_1$. At the second stage the temperature changes over the entire volume of the body $0 \le \xi \le 1$. Here an additional sought function $q_2(\mathrm{Fo}) = \Theta(1, \mathrm{Fo})$ characterizing the change with time in the temperature at the plate center is introduced.

The separation of the single process of heat conduction into two interrelated processes makes it possible to appreciably simplify the sequence of obtaining a solution of the problem, since in this case one can use, as it turns out, the method of approximation representation of the solution with determination of any number of its terms. To determine the unknown coefficients of the polynomial, it is necessary to use additional boundary conditions connected with the inclusion of boundary points on the spatial coordinate in the domain of definition of the input differential equation. The physical meaning of the additional boundary conditions is that their subjection to the sought solution is equivalent to the fulfillment of the input differential equation at the boundary points and at the temperature perturbation front (i.e., inside the domain). In so doing, the accuracy of satisfying the equation is fully determined by the number of additional boundary conditions on which the power of the approximating polynomial (the number of approximations) depends.

The mathematical statement of the boundary-value problem for the first stage of the process as a result of the introduction of the temperature perturbation front $q_1(\mathrm{Fo})$ will include Eq. (1) with the boundary condition (3), as well as the following boundary conditions fulfilled at the temperature perturbation front:

$$\Theta (q_1, \mathrm{Fo}) = 0 , \tag{5}$$

$$\partial \Theta (q_1, \mathrm{Fo}) / \partial \xi = 0 \quad (0 \le \xi \le q_1 (\mathrm{Fo})) , \tag{6}$$

where relations (5)–(6) represent the conditions of conjugation of the heated and unheated zones. Relation (5) sets the temperature of the body and the point $\xi = q_1(\mathrm{Fo})$ equal to its initial temperature. According to condition (6), the heat flow does not propagate beyond the temperature perturbation front (adiabatic wall condition). The mathematical proof of conditions (5) and (6) is given in [10].

Notice that at the first stage of the process, problem (1), (3), (5), (6) beyond the temperature perturbation front, i.e., in the section $q_1(\mathrm{Fo}) \le \xi \le 1$, has not been defined at all. Therefore, there is no need to fulfill the initial condition of the form (2) across the whole thickness of the plate (that is why such a condition is absent from the mathematical statement of problem (1), (3), (5), (6)). In the given case, it wholly suffices to satisfy the boundary condition (5), according to which for all $\xi = q_1(\mathrm{Fo})$ the temperature of the body is equal to its initial temperature. Moreover, the given problem does not require a boundary condition of the form (4) either, since it does not influence the heat transfer process at its first stage. All this makes it possible to markedly simplify both the process of obtaining an analytical solution of problem (1), (3), (5), (6) and the final expression for it as compared to the solution of problem (1)–(4) by the classical analytical methods.

In using a classical analytical solution obtained, e.g., by employing the method of separation of variables, the most difficult problems arise in the case of determining the temperature for small Fourier values ($\mathrm{Fo} \to 0$). This is due to the fact that the sought solution in this case should describe two practically straight lines of the temperature change AE and DC (see Fig. 1) connected to each other by an insignificant curvilinear segment ED and positioned almost at a right angle to each other. On one of these straight lines (DC line) the temperature is constant, and on the other line it changes practically from zero to unity with insignificant variations of the ξ coordinate. All these very contradictory and, what is more, time-variable conditions in one analytical expression (at $\mathrm{Fo} \to 0$) can only be satisfied by using in it an infinitely large number of series terms, as was mentioned above.

The simplicity of the method proposed here is that the solution obtained in the given case is not associated with the necessity of approximating the temperature on segments DC, BC, etc., i.e., after the temperature perturbation front $q_1(\mathrm{Fo}) \le \xi \le 1$, since in this region at the first stage of the process problem (1), (3), (5), (6) is not defined. The analytical solution here describes the change with time in the temperature characterized only by curves of the form of AD, AB. etc., for whose practically exact description it is enough to use only a few series terms of the solution whatever the value of the time coordinate.

Note that problem (1), (3), (5), (6) does not belong to the class of problems in which the finite thermal wave velocity is taken into account. Their solution is reduced to integration of the hyperbolic (wave) heat conduction equation [11]. The temperature perturbation front introduced in problem (1), (3), (5), (6) is analogous in physical meaning to the moving isotherm (but not the thermal wave). Since at the temperature perturbation front in the process of its motion along the ξ coordinate the initial temperature $\Theta q_1(\mathrm{Fo}) = 0$ is maintained, the front is an analog of the zero isotherm.

The solution of system (1), (3), (5), (6) is sought in the form of the following polynomial:

$$\Theta(\xi, \mathrm{Fo}) = \sum_{k=0}^{n} a_k(q_1)\, \xi^k , \tag{7}$$

where $a_k(q_1)$ denotes the unknown coefficients determined from the boundary conditions (3), (5), (6). Once $a_k(q_1)$ ($k = 0, 1, 2$) have been determined, relation (7) will take the form

$$\Theta(\xi, \mathrm{Fo}) = \left(1 - \frac{\xi}{q_1}\right)^2 . \tag{8}$$

To find the unknown function $q_1(\mathrm{Fo})$, we form the residual of Eq. (1) and determine the integral of it within the limits of the thermal layer depth (which is equivalent to the construction of the heat balance integral)

$$\int_0^{q_1(\mathrm{Fo})} \frac{\partial \Theta(\xi, \mathrm{Fo})}{\partial \mathrm{Fo}}\, d\xi = \int_0^{q_1(\mathrm{Fo})} \frac{\partial^2 \Theta(\xi, \mathrm{Fo})}{\partial \xi^2}\, d\xi . \tag{9}$$

Substituting (8) into (9), upon determining the integral we have

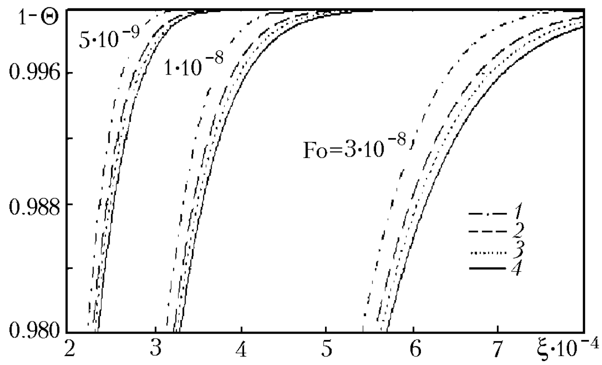

Fig. 2. Temperature change in the plate: 1) third approximation; 2) seventh approximation; 3) fourteenth approximation; 4) exact solution of [11].

$$q_1 dq_1 = 6d\text{Fo} . \tag{10}$$

Integrating (10), under the initial condition $q_1(0) = 0$ we get

$$q_1 = \sqrt{12\text{Fo}} . \tag{11}$$

Assuming $q_1(\text{Fo}_1) = 1$, from (11) we obtain the time of completion of the first stage of the process $\text{Fo}_1 = 1/12 = 0.08333$.

Relations (8), (11) define the solution of problem (1), (3), (5), (6) in the first approximation of the first stage of the process. The results of the calculations by formula (8) compared to the exact solution of [11] are presented in Fig. 2. Their analysis permits concluding that the discrepancy with the exact solution is 3–4%. The main error therewith arises because of the inexact fulfillment of the differential equation (1). Note that the boundary condition (3), the conditions at the temperature perturbation front (5), (6), as well as the heat balance integral (9) are fulfilled exactly.

An apparent way of increasing the accuracy of the solution is increasing the power of the approximating polynomial (7). To determine the unknown coefficients that appear thereby, it is necessary to use additional boundary conditions. To obtain them, let us differentiate sequentially the boundary conditions (3), (5), (6) with respect to the variable Fo and Eq. (1) with respect to the variable ξ. Comparing the thus obtained relations, we can determine the necessary number of additional boundary conditions. The sequence of their establishment is given in [8, 9]. For example, to find a solution in the second approximation, such additional boundary conditions are of the form

$$\frac{\partial^2 \Theta(0, \text{Fo})}{\partial \xi^2} = 0 , \quad \frac{\partial^2 \Theta(q_1, \text{Fo})}{\partial \xi^2} = 0 , \quad \frac{\partial^3 \Theta(q_1, \text{Fo})}{\partial \xi^3} = 0 . \tag{12}$$

In the second approximation, using the additional boundary conditions (12) jointly with given conditions (3), (5), (6), we can already determine six coefficients of relation (7) and give the temperature function in the form of a polynomial of power five. Substituting (7), restricting ourselves to six series terms, into all the above boundary conditions, for the unknown coefficients a_k ($k = \overline{0, 5}$), we obtain a system of six algebraic linear equations. Substituting the coefficients a_k found from the solution of this system into relation (7) yields

$$\Theta(\xi, \text{Fo}) = \left(1 + \frac{3}{2}\frac{\xi}{q_1}\right)\left(1 - \frac{\xi}{q_1}\right)^4 . \tag{13}$$

Substituting (13) into (9), for $q_1(\text{Fo})$ we arrive at the following ordinary differential equation:

$$q_1 dq_1 = 10d\text{Fo} . \tag{14}$$

Integrating (14), under the initial condition $q_1(0)$ we get

$$q_1 = \sqrt{20\mathrm{Fo}}\ . \tag{15}$$

Assuming in (15) $q_1(\mathrm{Fo}_1) = 1$, we find the time of completion of the first stage of the process in the second approximation $\mathrm{Fo} = \mathrm{Fo}_1 = 0.5$.

Relations (13), (15) define the solution of problem (1), (3), (5), (6) with the additional boundary conditions (12) in the second approximation. Comparison of the results of calculations by formula (13) with the exact solution of [11] makes it possible to draw the conclusion that in the range of Fourier numbers $1\cdot 10^{-5} \le \mathrm{Fo} \le \mathrm{Fo}_1 = 0.05$ their discrepancy does not exceed 1%.

The following three additional boundary conditions for obtaining a solution in the third approximation have the form

$$\frac{\partial^4 \Theta\,(0, \mathrm{Fo})}{\partial \xi^4} = 0\,, \quad \frac{\partial^4 \Theta\,(q_1, \mathrm{Fo})}{\partial \xi^4} = 0\,, \quad \frac{\partial^5 \Theta\,(q_1, \mathrm{Fo})}{\partial \xi^5} = 0\,. \tag{16}$$

The additional boundary conditions (12), (16) jointly with given conditions (3), (5), (6) permit finding the unknown coefficients a_k $(k = \overline{0, 8})$ from [7]. Substituting the values of the coefficients a_k determined from the solution of the corresponding system of algebraic equations into (7), we arrive at the following relation for determining the temperature in the third approximation:

$$\Theta\,(\xi, \mathrm{Fo}) = \left(1 + 3\frac{\xi}{q_1} + 3\frac{\xi^2}{q_1^2}\right)\left(1 - \frac{\xi}{q_1}\right)^6 . \tag{17}$$

Substituting (17) into (9) yields

$$\frac{5}{24}\frac{dq_1\,(\mathrm{Fo})}{d\mathrm{Fo}} = \frac{3}{q_1\,(\mathrm{Fo})}\,. \tag{18}$$

Integrating Eq. (18), under the initial condition $q_1(0) = 0$, we obtain

$$q_1\,(\mathrm{Fo}) = 12\,\sqrt{5\mathrm{Fo}}/5\,.$$

In this case, the time of completion of the third stage of the process determined from the condition $q_1(\mathrm{Fo}_1) = 1$ will be $\mathrm{Fo} = \mathrm{Fo}_1 = 0.03472$.

Analysis of the results obtained makes it possible to draw the conclusion that the ordinary differential equations with respect to the function $q_1(\mathrm{Fo})$ in any approximation are identical in form and differ only by the coefficients. Their integration presents no problems.

Likewise, we can also write the additional boundary conditions for any other approximations. In particular, solutions in the fourth, fifth, tenth, and fourteenth approximations have been obtained in this manner. For example, the solution of problem (1), (3), (5), (6) in the fifth approximation is written in the form of the following algebraic polynomial:

$$\Theta\,(\xi, \mathrm{Fo}) = \sum_{k=0}^{14} A_k \left(\frac{\xi}{q_1}\right)^k , \tag{19}$$

where $A_0 = 1$; $A_2 = A_4 = A_6 = A_8 = 0$; $A_1 = -245/64$; $A_3 = 455/32$; $A_5 = -3003/64$; $A_7 = 2145/16$; $A_9 = -35{,}035/64$; $A_{10} = 1001$; $A_{11} = -28{,}665/32$; $A_{12} = 455$; $A_{13} = -8085/64$; $A_{14} = 15$; $q_1(\mathrm{Fo}) = 2\sqrt{105\mathrm{Fo}}/3$.

Note that relation (19) represents an exponential algebraic polynomial with respect to the variables ξ and Fo containing neither special functions (Bessel, Legendre, gamma functions, etc.) nor trigonometric functions.

The results of the calculations for the 3d, 7th, and 14th approximations compared to the exact solution of [11] are given in Fig. 2. Their analysis leads to the conclusion that with increasing number of approximations the solution is corrected each time. For instance, already in the seventh approximation the values of temperatures in the range of numbers $5 \cdot 10^{-9} \leq \text{Fo} \leq \text{Fo}_1$ differ from their exact values by no more than 0.002%, and in the fourteenth approximation — by 0.0004%. It should be noted that it is difficult to obtain an exact solution by the formulas from [11] for such small Fourier numbers because of the necessity of using a large number of series terms of the exact solution. In particular, calculations have shown that at Fo = 10^{-7}, for the exact solution to converge, one has to use about 1000 series terms (see formula (16) on page 87 in [11]). For numbers Fo = 10^{-8}, 10^{-9}, 10^{-10}, 10^{-11}, and 10^{-12}, convergence of the exact solution is observed, respectively, at the following values of the series numbers: 5000, 10,000, 50,000, 200,000, and 500,000.

Analysis of the results obtained makes it possible to conclude that the classical methods of linear superposition of particular solutions used to fulfill the initial condition have low efficiency. Precisely at this stage of obtaining the classical solution the maximum complication occurs, that is due to the necessity of subjecting the solution to the above-mentioned highly contradictory conditions that turn out to be practically impossible to fulfill at Fo $\to$ 0 (Fo $\neq$ 0) because of the necessity of using in the classical solution an infinite number of series terms. Representing the initial boundary-value problem in the form of two interrelated processes considered separately and related only by the conjugation condition at Fo = Fo_1 permits avoiding the above difficulties with the possibility of obtaining a solution with practically any degree of accuracy.

The accuracy of the solution increases with increasing number of approximations due to the increase in the accuracy of the fulfillment of Eq. (1), which is confirmed by the analysis of the change in its residual. For example, the maximum residual in the seventh approximation is ε = 0.054, and in the fourteenth approximation it decreases to ε = 0.005.

The method of additional boundary conditions can also be used for the second stage of the heating (cooling) process. It corresponds to the time $\text{Fo} \geq \text{Fo}_1$ and is characterized by a temperature variation already over the entire cross-section of the plate up to the moment the steady state is reached. For this stage, the notion of the thermal layer loses sense, and, as an additional thought function, the function $\Theta(1, \text{Fo}) = q_2(\text{Fo})$ characterizing the temperature change depending on the time at the center of the plate is taken (Fig. 1).

The mathematical statement of the problem for the second stage of the thermal process includes Eq. (1) with the boundary condition (3), to which two more boundary conditions

$$\Theta(1, \text{Fo}) = q_2(\text{Fo}) \,, \tag{20}$$

$$\partial\Theta(1, \text{Fo})/\partial\xi = 0 \tag{21}$$

are added.

Problem (1), (3), (20), (21) contains no boundary condition for the following reasons. At Fo = Fo_1, q_1(Fo1) = 1 and $q_2(\text{Fo}_1) = 0$. In this case, the boundary conditions (20), (21) become identical to the boundary conditions (5), (6). Consequently, at Fo = Fo_1 the mathematical statements of problems (1), (3), (5), (6) and (1), (3), (20), (21) fully coincide. Thus there occurs a smooth transition from the first stage of the process to the second one. Since, at Fo = Fo_1, $q_1(\text{Fo}_1) = 1$, relation (8) will have the form

$$\Theta(\xi, \text{Fo}_1) = (1 - \xi)^2 \,. \tag{22}$$

Relation (22) represents the initial condition of problem (1), (3), (20), (21). However, we need not satisfy it specially: it will be fulfilled in solving problem (1), (3), (20), (21).

As in the first stage, a solution to problem (1), (3), (20), (21) is sought in the form of a polynomial to the nth power

$$\Theta(\xi, \text{Fo}) = \sum_{k=0}^{n} b_k(q_2)\, \xi^k \,. \tag{23}$$

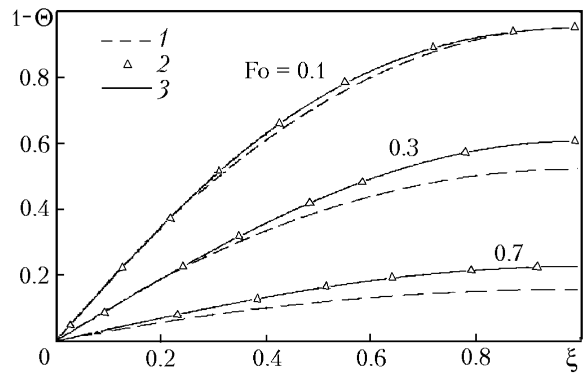

Fig. 3. Change in the relative excess temperature in the second stage of the process: 1) by formula (28) (first approximation); 2) by formula (33) (second approximation); 3) exact solution of [11].

The unknown coefficients b_k (k = 0, 1, 2) are found from the boundary conditions (3), (20), (21). Having determined them and substituted into (23) we will have

$$\Theta (\xi, \mathrm{Fo}) = 1 - (1 - q_2)\, \xi\, (2 - \xi)\,. \tag{24}$$

To obtain a solution in the first approximation, let us form the residual of the differential equation (1) and integrate it in the limit from $\xi = 0$ to $\xi = 1$, i.e.,

$$\int_0^1 \frac{\partial \Theta (\xi, \mathrm{Fo})}{\partial \mathrm{Fo}}\, d\xi = \int_0^1 \frac{\partial^2 \Theta (\xi, \mathrm{Fo})}{\partial \xi^2}\, d\xi\,. \tag{25}$$

Substituting (24) into (25) and defining the integrals, for the unknown function q_2(Fo) we arrive at the following ordinary differential equation:

$$\frac{\partial q_2 (\mathrm{Fo})}{\partial \mathrm{Fo}} + 3 q_2 (\mathrm{Fo}) - 3 = 0\,. \tag{26}$$

Integrating Eq. (26), under the initial condition $q_2(\mathrm{Fo}_1) = 0$ we obtain

$$q_2 (\mathrm{Fo}) = 1 - \exp\, [- 3\, (\mathrm{Fo} - \mathrm{Fo}_1)]\,. \tag{27}$$

Substituting (27) into (24) yields

$$\Theta (\xi, \mathrm{Fo}) = 1 - \xi\, (2 - \xi) \exp\, [- 3\, (\mathrm{Fo} - \mathrm{Fo}_1)]\,, \tag{28}$$

where $\mathrm{Fo}_1 = 0.0833$ was found from relation (11).

The results of the calculations by formula (28) compared to the exact solution of [11] are given in Fig. 3. Their analysis permits concluding that the maximum difference of the temperatures obtained by formula (28) from their exact values constitutes 8%. Solution (28) exactly satisfies the initial condition (22) and the boundary conditions (3), (20), (21). Consequently, the total solution error is due to the inexact fulfillment of the differential equation (1). Indeed, as follows from relation (25), Eq. (1) is satisfied only in the plate thickness on average.

The increase in the accuracy of the solution is due to the increase in the number of series terms (23). The unknown coefficients that appear thereby can be found from the additional boundary conditions. To determine them, we use the initial boundary conditions (3), (20), (21) which are differential with respect to the variable Fo, as well as Eq. (1) which is differentiated with respect to the variable ξ. Comparing the relations obtained thereby, we get the following three additional boundary conditions [8, 9]:

$$\frac{\partial^2\Theta(0,\mathrm{Fo})}{\partial\xi^2}=0\,,\quad \frac{\partial^2\Theta(1,\mathrm{Fo})}{\partial\xi^2}=\frac{dq_2(\mathrm{Fo})}{\partial\mathrm{Fo}}\,,\quad \frac{\partial^3\Theta(1,\mathrm{Fo})}{\partial\xi^3}=0\,. \tag{29}$$

Using the basic (3), (20), (21) and additional (29) boundary conditions, we can already determine six coefficients of series (23). Substituting (23) in all the above boundary conditions, for the unknown coefficients b_k ($k = \overline{0, 5}$) we obtain a system of six algebraic linear equations. Determining from the solution of this system the coefficients b_k and substituting them into (23), we find

$$\Theta(\xi,\mathrm{Fo})=1-\frac{1}{2}(5\xi-10\xi^3+10\xi^4-3\xi^5)(1-q_2)-\frac{1}{8}(3\xi-14\xi^3+16\xi^4-5\xi^5)\frac{dq_2}{d\mathrm{Fo}}\,. \tag{30}$$

Substituting (30 into (25), for determining the unknown function $q_2(\mathrm{Fo})$, we have the following second-order inhomogeneous ordinary differential equation:

$$\frac{11}{240}\frac{d^2q_2}{d\mathrm{Fo}^2}+\frac{9}{8}\frac{dq_2}{d\mathrm{Fo}}+\frac{5}{2}q_2-\frac{5}{2}=0\,. \tag{31}$$

The general solution of Eq. (31) is of the form

$$q_2(\mathrm{Fo})=1+C_1\exp(-2.4709\mathrm{Fo})+C_2\exp(-22.0745\mathrm{Fo})\,.$$

The integration constants C_1 and C_2 are found from the initial conditions $q_2(\mathrm{Fo}) = 0$; $dq_2(\mathrm{Fo}_1)/d\mathrm{Fo} = 0$. Once they have been determined, the formula for $q_2(\mathrm{Fo})$ will be

$$q_2(\mathrm{Fo})=1-1.1261\exp[-2.4709(\mathrm{Fo}-\mathrm{Fo}_1)]+0.1261\exp[-22.0745(\mathrm{Fo}-\mathrm{Fo}_1)]\,. \tag{32}$$

Substituting (32) into (30), we find the final expression for solving problem (1), (3), (20), (21) in the second approximation of the second stage of the process

$$\Theta(\xi,\mathrm{Fo})=1-(1.772\xi-0.761\xi^3+0.065\xi^4+0.05\xi^5)\exp[-2.471(\mathrm{Fo}-\mathrm{Fo}_1)]$$
$$-(0.728\xi-4.239\xi^3+4.935\xi^4-1.55\xi^5)\exp[-22.074(\mathrm{Fo}-\mathrm{Fo}_1)]\,, \tag{33}$$

where $\mathrm{Fo}_1 = 0.05$ (found in the second approximation of the first stage of the process).

The results of the calculations of dimensionless temperatures by formula (33) compared to the exact solution of [11] are presented in Fig. 3. Their analysis permits concluding that the solution obtained here throughout the range of change in the Fourier number of the second stage of the process practically coincides with the exact one. Note that the coefficients positioned under the exponent sign differ insignificantly from the first two eigenvalues of the Sturm–Liouville boundary-value problem, whose exact values are $\mu_1 = 2.4674$ and $\mu_2 = 22.2066$.

Because of the fairly high accuracy of the solution in the second approximation of the second stage of the process, the fulfillment of the third approximation is only of theoretical interest. For obtaining a solution to problem (1), (3), (20), (21) in the third approximation, the following three additional boundary conditions have the form [8, 9]

$$\frac{\partial^4\Theta(0,\mathrm{Fo})}{\partial\xi^4}=0\,,\quad \frac{\partial^4\Theta(1,\mathrm{Fo})}{\partial\xi^4}=\frac{d^2q_2}{\partial\mathrm{Fo}^2}\,,\quad \frac{\partial^5\Theta(1,\mathrm{Fo})}{\partial\xi^5}=0\,. \tag{34}$$

Using the main boundary conditions (3), (20), (21) and the additional boundary conditions (34), we can already determine the coefficients of polynomials (23) and find the temperature function in the third approximation.

A characteristic feature of the analytical solutions obtained here is the polynomial dependence of temperature on the ξ coordinate as opposed to the classical exact solutions where they are related through the trigonometric func-

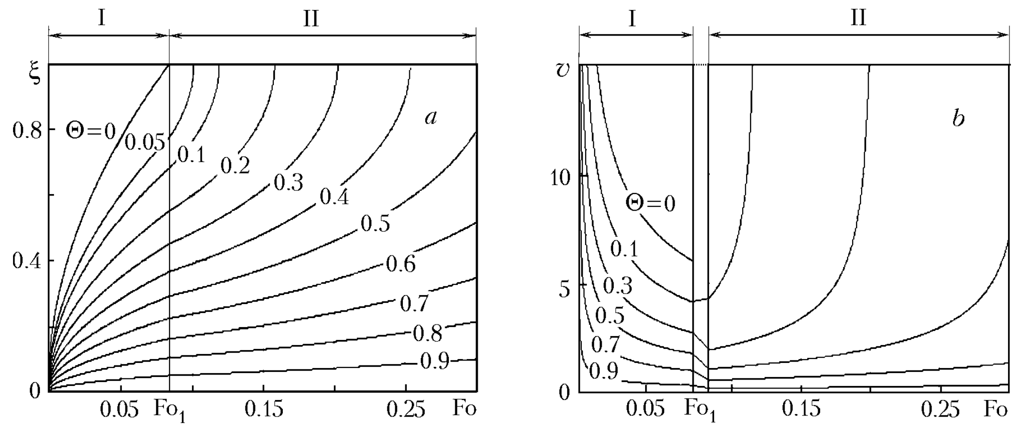

Fig. 4. Distribution of isotherms (a) and velocities of their motion (b) in the plate at boundary conditions of the first kind. I, II, stages of the process.

tions. The polynomial dependence permits obtaining a solution in the form of the field of isothermal lines. Note that the obtaining of isotherms on the basis of the classical analytical solutions was considered in [12].

Consider the principle of construction of isotherms with the example of the first approximation of the first and second stages of the process. Expressing the ξ coordinate as a function of the temperature Θ(ξ, Fo) and the time Fo, let us reduce relations (8) and (28) to the form

$$\xi = \left(1 - \sqrt{\Theta\,(\xi, \mathrm{Fo})}\right) \sqrt{12\mathrm{Fo}}\,, \tag{35}$$

$$\xi = E - \sqrt{E\,(E - 1 + \Theta)}/E\,. \tag{36}$$

Relations (35), (36) permit, for any concrete Θ(ξ, Fo) = const, plotting curves of temperature versus ξ and Fo (isotherm curves) (see Fig. 4a). Note that the zero isotherm Θ(ξ, Fo) = 0 coincides with the curve of motion of the temperature perturbation front along the ξ coordinate depending on the time Fo. Indeed, at Θ(ξ, Fo) = 0 expression (35) takes on the form of the expression $\xi = \sqrt{12\mathrm{Fo}}$ completely coinciding with formula (11) characterizing the travel of the perturbation temperature front. From this it follows that, as to its physical meaning, the perturbation temperature front is an analog of the isotherm moving with time along the ξ coordinate. In the given case, this is a zero isotherm — the isotherm of the initial condition.

The first time derivatives of relations (35), (36) enable us to determine the dimensionless velocities of motion of isotherms $v = d\xi/d\mathrm{Fo}$ along the ξ coordinate depending on time, and the second derivatives permit determining the accelerations $a = d^2\xi/d\mathrm{Fo}^2$. The formulas for the velocities of the first and second stages of the process will be

$$v = -\sqrt{3/\mathrm{Fo}}\,(\sqrt{\Theta} - 1)\,, \tag{37}$$

$$v = 3\,(1 - \Theta)/\left[2\,\sqrt{E\,(E - 1 + \Theta)}\right], \tag{38}$$

respectively.

The velocity-time curves of isotherms found by formulas (37), (38) are given in Fig. 4b. Their analysis permits concluding that isotherms have the maximum velocities in the vicinity of the point ξ = 0. Upon receding from, it the velocities decrease appreciably, reaching some minimum. Then, with approach to the point ξ = 1, the velocities increase appreciably again. The acceleration-time curves practically coincide in shape with the velocity-time curves and differ from them only quantitatively.

Because of the low accuracy of the first approximation (and especially at the second stage of the process), the isotherms determined by formulas (35), (36) have a small kink at $\mathrm{Fo} = \mathrm{Fo}_1 = 0.0833$, i.e., at the point of conjugation of the solutions for the first and second stages of the process (see Fig. 4a). In this connection, on the curves of Fig. 4b there is some jump in the velocity diagrams, which already in the second approximation is practically absent, as is the kink in the isotherms.

To construct mathematical lines in the ξ–Fo coordinates as applied to the following approximations, for each concrete Θ(ξ, Fo) and Fo with respect to ξ, one has to solve the algebraic polynomial. Since to each ξ and Fo, according to the analytical solutions obtained, there corresponds only the value of the temperature Θ(ξ, Fo), then the algebraic polynomial has only one root satisfying the corresponding solutions of the form (13), (17), (19), (33). For example, in the second approximation of the first stage of the process the polynomial algebraic over ξ obtained from relation (13) at Θ = 0.5 and Fo = 0.05 is of the form

$$0.5 - 2.5\xi + 5\xi^3 - 5\xi^4 + 1.5\xi^5 = 0 \,. \tag{39}$$

To obtain a polynomial algebraic over the time variable ξ in the second approximation of the second stage of the process, formula (33) is used. For example, at Θ = 0.5 and Fo = 0.2 it will be

$$0.5 - 1.2427\xi + 0.6799\xi^3 - 0.2249\xi^4 + 0.022\xi^5 = 0 \,. \tag{40}$$

Of the five roots of polynomial (39), the only solution satisfying relation (13) at Θ = 0.5 and Fo = 0.05 is the value of ξ = 0.2161.

Likewise, the only solution of polynomial (40) satisfying relation (33) at Θ = 0.5 and Fo = 0.2 is the root ξ = 0.4399.

Using the above method, we have obtained analytical solutions to the following heat-conduction problems: for a cylinder and a sphere under boundary conditions of the first kind; for a plate, a cylinder, and a sphere under boundary conditions of the third kind; for time-variable boundary conditions of the first kind; for boundary conditions of the third kind with a time-variable temperature of the medium and heat conductivity coefficients; for time-variable boundary conditions of the second kind; for time-variable internal heat-sources; and under a variable initial condition. We have also solved nonlinear heat conduction problems and problems of heat transfer in a laminar fluid flow in a plane-parallel channel, and other problems. The analytical solutions of some of these problems, as well as the distribution curves of isotherms and their velocities, are given below.

The analytical solution of the heat conduction problem for a cylinder (sphere) under boundary conditions of the first kind in the second approximation of the first stage of the process is of the form

$$\Theta(\xi, \mathrm{Fo}) = 1 - \frac{20}{cq_1 + 8}\frac{\xi}{q_1} - \frac{10cq_1}{cq_1 + 8}\left(\frac{\xi}{q_1}\right)^2 + \frac{20\,(cq_1 + 2)}{cq_1 + 8}\left(\frac{\xi}{q_1}\right)^3$$
$$- \frac{5\,(3cq_1 + 8)}{cq_1 + 8}\left(\frac{\xi}{q_1}\right)^4 + \frac{4\,(cq_1 + 3)}{cq_1 + 8}\left(\frac{\xi}{q_1}\right)^5 \quad (0 < \mathrm{Fo} \le \mathrm{Fo}_1)\,, \tag{41}$$

where c = 1, 2 for the cylinder and the sphere, respectively; $q_1(\mathrm{Fo}) = 5.53918\ \mathrm{Fo}^{0.54}$ (at c = 1); $\mathrm{Fo}_1 = 0.042$.

In the second approximation of the second stage of the process, the analytical solution for the cylinder is written in the form of the formula

$$\Theta(\xi, \mathrm{Fo}) = 1 - (1.56\xi + 0.78\xi^2 - 1.071\xi^3 - 0.221\xi^4 + 0.195\xi^5)\exp[-5.805\,(\mathrm{Fo} - \mathrm{Fo}_1)]$$
$$- (0.662\xi + 0.331\xi^2 - 5.596\xi^3 + 6.332\xi^4 - 1.973\xi^5)\exp[-29.682\,(\mathrm{Fo} - \mathrm{Fo}_1)] \quad (\mathrm{Fo}_1 \le \mathrm{Fo} < \infty)\,. \tag{42}$$

The curves of isotherms and their velocities for the cylinder and the sphere are analogous qualitatively to the corresponding curves for the plate and differ from them only quantitatively. Of the three bodies of classical form (plate, cylinder, sphere) the highest velocities of isotherms are observed in the sphere. The time of completion of the first stage of the process in the sphere therewith turns out to be the smallest and in the plate — the largest.

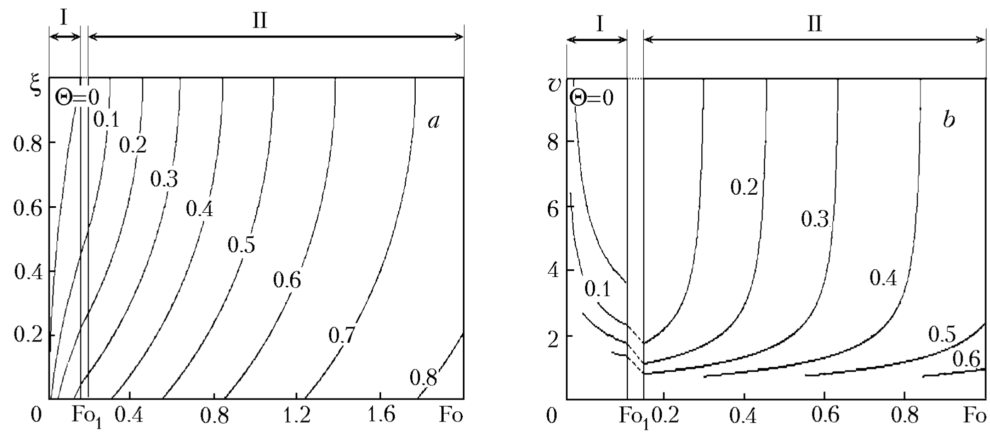

Fig. 5. Distribution of isotherms (a) and velocities of their motion (b) in the plate at boundary conditions of the third kind (Bi = 1). I, II, stages of the process.

In the case of an infinite plate under boundary conditions of the third kind, the solution in the first approximation of the first stage of the process is written in the form

$$\Theta(\xi, \text{Fo}) = \frac{\text{Bi}}{2 + \text{Bi}\, q_1}\left(q_1 - 2\xi + \frac{\xi^2}{q_1}\right) \quad (0 < \text{Fo} \le \text{Fo}_1)\,, \tag{43}$$

where $q_1(\text{Fo}) = 2.7163\ \text{Fo}^{0.52}$ (at Bi = 1).

In the first approximation of the second stage, we have

$$\Theta(\xi, \text{Fo}) = 1 - \frac{1}{2 + \text{Bi}} \exp\left[-\frac{3\text{Bi}\,(\text{Fo} - \text{Fo}_1)}{3 + \text{Bi}}\right](2 + 2\text{Bi}\,\xi - \text{Bi}\,\xi^2) \quad (\text{Fo}_1 \le \text{Fo} < \infty)\,. \tag{44}$$

The distribution curves of isotherms and velocities of their motion, e.g., for Bi = 1, are given in Fig. 5. Their analysis permits concluding that each isotherm appears on the plate surface at a strictly definite instant of time with a concrete initial velocity.

In the case of time-variable boundary conditions of the first kind (the wall temperature is a linear function of time $T(R, \tau) = T_0 + b\tau$), the analytical solution in the second approximation of the first stage of the process has the form

$$\frac{\Theta(\xi, \text{Fo})}{\text{Pd}} = \text{Fo} - \frac{20\text{Fo} + q_1^2}{8q_1}\xi + \frac{1}{2}\xi^2 + \frac{20\text{Fo} - 3q_1^2}{4q_1^3}\xi^3 - \frac{10\text{Fo} - q_1^2}{2q_1^4}\xi^4 + \frac{12\text{Fo} - q_1^2}{8q_1^5}\xi^5 \quad (0 < \text{Fo} \le \text{Fo}_1)\,, \tag{45}$$

where $q_1(\text{Fo}) = \sqrt{20\text{Fo}}$; $\text{Fo}_1 = 0.05$.

In the second approximation of the second stage of the process, we give the formula for the dimensionless temperature as

$$\frac{\Theta(\xi, \text{Fo})}{\text{Pd}} = \text{Fo} - \xi + 0.5\xi^2 + (0.717\xi - 0.308\xi^3 + 0.0265\xi^4 + 0.0202\xi^5)$$

$$\times \exp[-2.471\,(\text{Fo} - \text{Fo}_1)] + (0.033\xi - 0.192\xi^3 + 0.223\xi^4 - 0.0702\xi^5)$$

$$\times \exp[-22.075\,(\text{Fo} - \text{Fo}_1)] \quad (\text{Fo}_1 \le \text{Fo} < \infty)\,. \tag{46}$$

The distribution curves of isotherms and velocities of their motion in the case of time-variable boundary conditions of the first kind are given in Fig. 6.

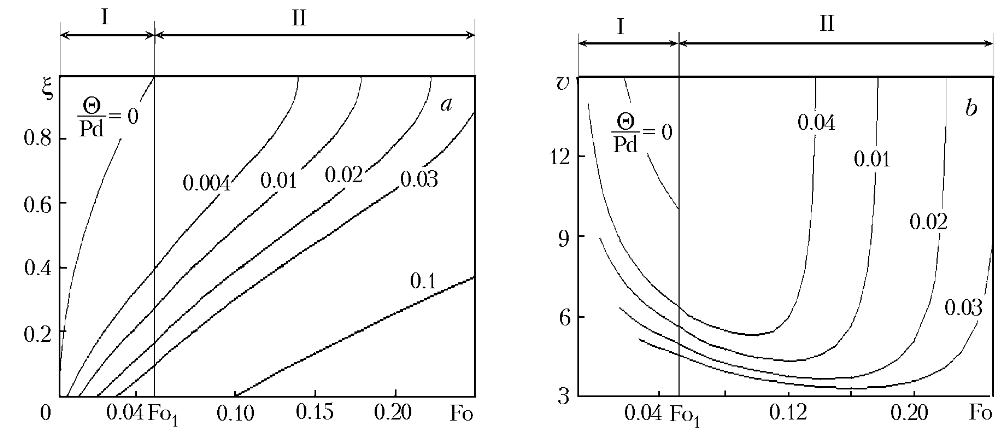

Fig. 6. Distribution of isotherms (a) and velocities of their motion (b) in the plate at boundary conditions of the first kind with a time-variable temperature of the wall. I, II, stages of the process.

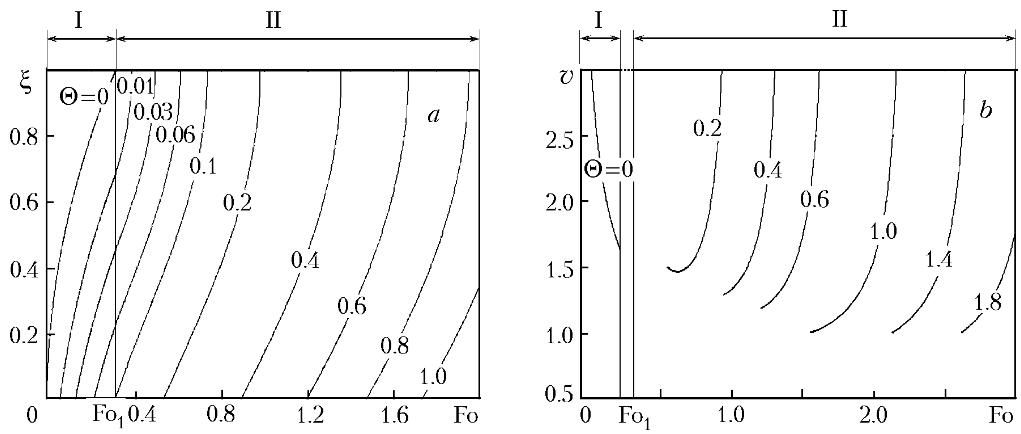

Fig. 7. Distribution of isotherms (a) and velocities of their motion (b) in the plate at boundary conditions of the third kind with a time-variable temperature of the medium (Bi = 1, Pd = 1). I, II, stages of the process.

The analytical solutions of the heat conduction equation for an infinite plate at boundary conditions of the third kind with a time-variable temperature of the medium (the temperature of the medium is a linear function of time $T_m(\tau) = T_0 + b\tau$) in the first approximation for the first and second stages of the process, respectively, have the form

$$\Theta_1(\xi, \mathrm{Fo}) = \frac{\mathrm{Bi\,Pd\,Fo}}{2 + \mathrm{Bi}\,q_1}\left(q_1 - 2\xi + \frac{\xi^2}{q_1^2}\right) \quad (0 < \mathrm{Fo} \le \mathrm{Fo}_1)\,, \tag{47}$$

$$\Theta_2(\xi, \mathrm{Fo}) = \frac{1}{1 + \mathrm{Bi}}\left[\mathrm{Bi\,Pd\,Fo} + 2q_2 + \mathrm{Bi}\,\xi\,(q_2 - \mathrm{Pd\,Fo})\,(2 - \xi)\right] \quad (\mathrm{Fo}_1 \le \mathrm{Fo} < \infty)\,, \tag{48}$$

where $q_1(\mathrm{Fo}) = 1.797\sqrt{\mathrm{Fo}}$; $q_2(\mathrm{Fo}) = \{(l + mc)\exp[m(\mathrm{Fo} - \mathrm{Fo}_1)] + lm\mathrm{Fo}_1 \exp[m(\mathrm{Fo} - \mathrm{Fo}_1)] - l - mc - lm\mathrm{Fo}\}/m^2$; $l = 6\mathrm{Bi\,Pd}/\eta$; $m = -6\mathrm{Bi}/\eta$; $c = -6\mathrm{Bi\,Pd}/\eta$; $\eta = 6 + 2\mathrm{Bi}$.

Analysis of the distribution of isotherms at a linear change in the temperature of the medium shows that each isotherm appears on the plate surface at a strictly definite instant of time (see Fig. 7a). In so doing, for the isotherm $0 \le \Theta \le 1.4$ periods determining the appearance of each subsequent isotherm (after equal potentials) turn out to be different. For $1.4 \le \Theta \le \infty$, the appearance of each subsequent isotherm occurs at regular intervals.

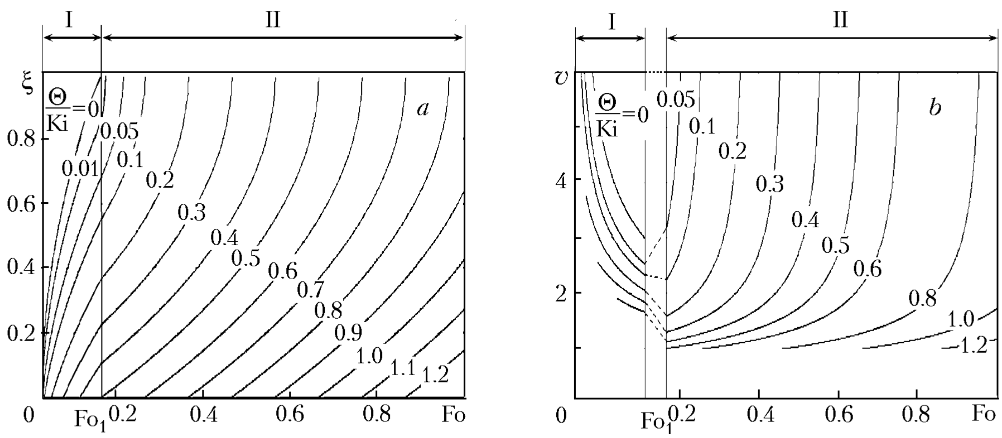

Fig. 8. Distribution of isotherms (Θ/Ki) (a) and velocities of their motion (b) in the plate at boundary conditions of the second kind. I, II, stages of the process.

Analysis of the distribution of velocities of motion of isotherms permits concluding that for $\Theta \geq 1.0$ the initial velocities of isotherms become equal and are $v_0 = 1.0$ (see Fig. 7b).

In the case of boundary conditions of the third kind with a time-variable heat transfer coefficient (the heat transfer coefficient is a linear function of time), the solutions for the first and second stages of the process in the first approximation have the form (for an infinite plate)

$$\Theta_1 (\xi, \text{Fo}) = \frac{\text{Bi}\,(1 + \text{Pd Fo})\left(q_1 - 2\xi + \dfrac{\xi^2}{q_1}\right)}{\text{Bi}\, q_1\, (1 + \text{Pd Fo}) + 2}, \tag{49}$$

$$\Theta_2 (\xi, \text{Fo}) = \frac{\text{Bi}\,[1 + \text{Pd Fo}_1 + (q_2 - 1)\,(1 + \text{Pd Fo}_1)\,(2\xi - \xi^2)] + 2q_2}{2 + \text{Bi}\,(1 + \text{Pd Fo})}, \tag{50}$$

where

$$q_1 (\text{Fo}) = 2.558\text{Fo}^{0.51}\,; \quad q_2 (\text{Fo}) = 1 - \exp\left[\ln \frac{3 + \text{Bi}\,(1 + \text{Pd Fo}_1)}{3 + \text{Bi}\,(1 + \text{Pd Fo})}\right.$$

$$\left. - \ln \frac{2 + \text{Bi}\,(1 + \text{Pd Fo}_1)}{2 + \text{Bi}\,(1 + \text{Pd Fo})} - \frac{9}{\text{Bi Pd}} \ln \frac{3 + \text{Bi}\,(1 + \text{Pd Fo}_1)}{3 + \text{Bi}\,(1 + \text{Pd Fo})} - 3\,(\text{Fo} - \text{Fo}_1)\right].$$

Analytical solutions of the heat conduction problem for the plate under time-variable boundary conditions of the second kind (the heat flow is a linear function of time) in the first approximation of the first and second stages of the process are written in the form of the following relations:

$$\Theta (\xi, \text{Fo}) = 0.5\text{Ki}(\text{Fo})\left(0.5q_1 - \xi + \frac{0.5\xi^2}{q_1}\right), \tag{51}$$

$$\Theta (\xi, \text{Fo}) = q_2 (\text{Fo}) + \text{Ki}\,(\text{Fo})\,(0.5 - \xi + 0.5\xi^2)\,, \tag{52}$$

where $q_1(\text{Fo}) = \sqrt{3\text{Fo}}$; $q_2(\text{Fo}) = \text{Ki}(\text{Fo})(\text{Fo} - \text{Fo}_1)$; $\text{Ki}(\text{Fo}) = \text{Ki}_0\text{Fo}$.

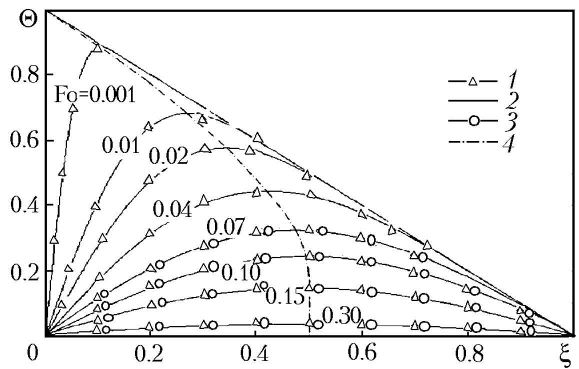

Fig. 9. Temperature distribution in the plate at a variable condition: 1) sweep method; 2) by formula (53) (in the range $0.001 \le \mathrm{Fo} \le 0.05$; first stage, second approximation); 3) by formula (54) (in the range $0.001 \le \mathrm{Fo} < \infty$; second stage, second approximation); 4) local symmetry line of the temperature — absence of heat exchange.

The distribution curves of isotherms and velocities of their motion in the case of a constant heat flow (Ki = const) are given in Fig. 8.

The heat conductivity problem for an infinite plate under a variable initial condition (the initial temperature is a linear function of the spatial coordinate) in the second approximation of the first and second stages of the process has the following solutions:

$$\Theta_1 (\xi, \mathrm{Fo}) = 1 - \xi - \left(1 + \frac{3}{2}\frac{\xi}{q_1}\right)\left(1 - \frac{\xi}{q_1}\right)^4 \quad (0 < \mathrm{Fo} \le \mathrm{Fo}_1) , \tag{53}$$

$$\Theta_2 (\xi, \mathrm{Fo}) = (1.3\xi - 3\xi^3 + 2\xi^4 - 0.3\xi^5) \exp [10 (\mathrm{Fo}_1 - \mathrm{Fo})]$$

$$- (0.8\xi - 0.4\xi^3 + 5\xi^4 - 1.8\xi^5) \exp [60 (\mathrm{Fo}_1 - \mathrm{Fo})] \quad (\mathrm{Fo}_1 \le \mathrm{Fo} < \infty) , \tag{54}$$

where $q_1(\mathrm{Fo}) = \sqrt{20\mathrm{Fo}}$; $\mathrm{Fo}_1 = 0.05$.

The results of calculations of the dimensionless temperature by formulas (53), (54) compared to the calculation by the finite difference method (sweep method), as well as the distribution curves of isotherms, are given in Figs. 9, 10a.

Analysis of the results obtained permits concluding that isotherms of one and the same potential arise both at the point $\xi = 0$ and on the line of propagation of the perturbation temperature front $q_1(\mathrm{Fo})$ and move in opposite directions. Thus, isotherms arising at the point $\xi = 0$ move in the direction of the ξ axis, and isotherms that appear on the line of $q_1(\mathrm{Fo})$ move in the direction opposite to this axis. Isotherms of one and the same potential are connected on the line where the condition of the absence of heat exchange is fulfilled (the line of local symmetry of the temperature field). In Fig. 10a the symmetry line passes through the points of maximal temperatures taking place in the cooling process (peaks of the temperature curves).

Analysis of the time distribution of velocities of motion of isotherms along the ξ coordinate (Fig. 10b) leads to the conclusion that the initial velocities of isotherms arising at the point $\xi = 0$ tend to infinite values. Then on a relatively short period the velocities decrease to certain minimum values. As the axial line of the absence of heat exchange is approached, the velocities of motion of isotherms increase again to infinite values.

All isotherms arising on the line of the perturbation temperature front $q_1(\mathrm{Fo})$ have zero initial velocities. Then, as the axial line of the absence of heat exchange is approached, the velocities of motion of isotherms tend to infinite values. Since isotherms arising on the $q_1(\mathrm{Fo})$ line move in the direction opposite to the ξ-axis, their velocities on the curves of Fig. 10b are taken conventionally with the minus sign.

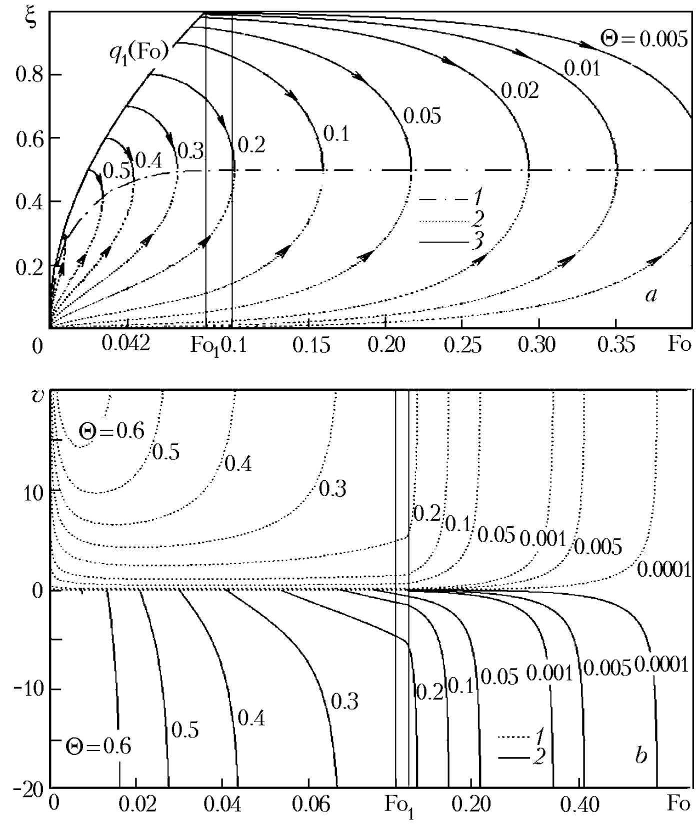

Fig. 10. Distribution of isotherms (a) and velocities of their motion (b) in the plate at a variable initial condition: a) 1) line of absence of heat exchange; 2) isotherms moving in the direction of the ξ axis; 3) isotherms moving in the direction opposite to the direction of the ξ axis; b) 1) change in the velocities of isotherms moving in the direction of the ξ axis; 2) in the opposite direction.

The analytical solution to the heat conduction equation for an infinite plate with an internal heat source in the second approximation of the first and second stages of the process is written in the form of the following relations:

$$\Theta(\xi, \text{Fo}) = 1 + [20(Q-1)\,\text{Po}\,q_1^2]\frac{\xi}{8q_1} - 0.5\text{Po}\,\xi^2 - [20(Q-1) - 3\text{Po}\,q_1^2]\frac{\xi^3}{4q_1^3}$$

$$+ [40(Q-1) - 4\text{Po}\,q_1^2]\frac{\xi^4}{8q_1^4} + [12(1-Q) + \text{Po}\,q_1^2]\frac{\xi^5}{8q_1^5} \quad (0 < \text{Fo} \le \text{Fo}_1)\,, \tag{55}$$

$$\Theta(\xi, \text{Fo}) = 1 + \xi(\text{Po} - 0.5\xi) + A_1(1.5733\xi + 0.6756\xi^3 + 0.0578\xi^4 + 0.0444\xi^5)$$

$$\times \exp[2.4711(\text{Fo}_1 - \text{Fo})] + A_2(5.7781\xi + 33.631\xi^3 - 39.15\xi^4 + 12.297\xi^5)$$

$$\times \exp[22.075(\text{Fo}_1 - \text{Fo})] \quad (\text{Fo}_1 \le \text{Fo} < \infty)\,, \tag{56}$$

where $q_1(\text{Fo}) = \sqrt{20\text{Fo}}$; $Q = \text{PoFo}$; $A_1 = -0.512\text{Po} - 1.1261 + 1.1261\text{PoFo}_1$; $A_2 = 0.012\text{Po} + 0.12605 - 0.12605\text{PoFo}_1$.

The results of calculations of the dimensionless temperature $\Theta = (T - T_0)/(T_w - T_0)$ by formulas (55), (56), as well as the distribution curves of isotherms, are given in Figs. 11 and 12.

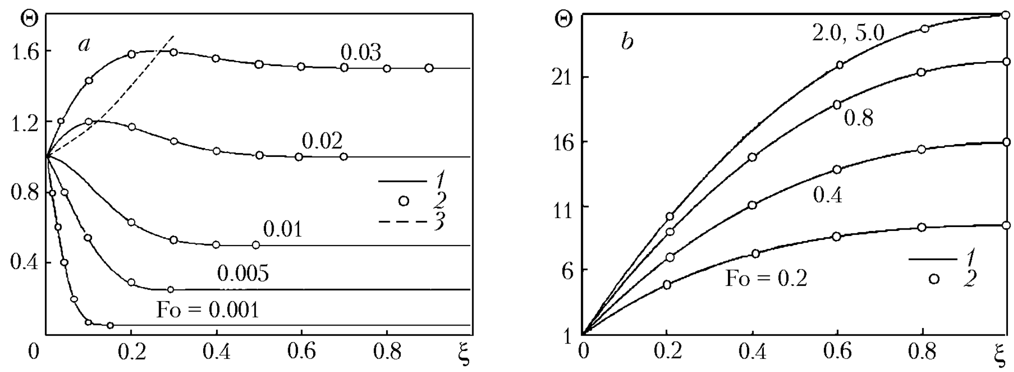

Fig. 11. Temperature distribution in the plate with an internal heat source (Po = 50): a) 1) exact solution of [11]; 2) by formula (55); 3) local symmetry line of the temperature field (absence of heat exchange); b) 1) exact solution of [11]; 2) by formula (56).

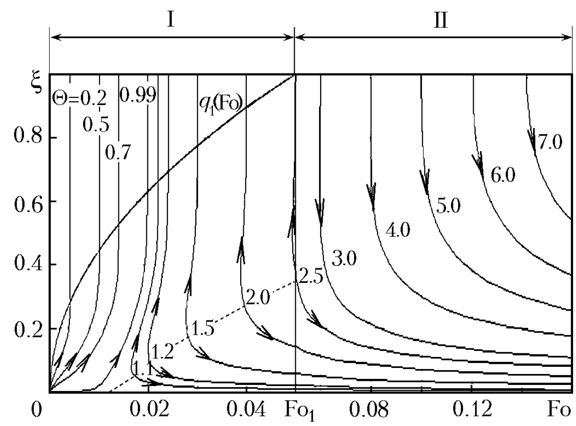

Fig. 12. Distribution of isotherms in the plate with an internal heat source (Po = 50). Dashed line shows the local symmetry line of the temperature field. I, II, stages of the process.

Analysis of the distribution of isotherms (Fig. 12) makes it possible to draw the following conclusions. At the first stage of the process, isotherms in the region before the perturbation temperature front have the form of curved lines. In the region after the perturbation temperature front, they degenerate into straight lines, which is due to the action of the heat source. For isotherms $1 < \Theta \leq 3.0$, local symmetry of the temperature caused by the mutual action of the boundary condition of the first kind and the heat source is observed (curve 3 in Fig. 11a). Isotherms of equal potential arising on the symmetry line move in opposite directions. As soon as the perturbation temperature front reaches the coordinate $\xi = 1$, the direction of motion of all isotherms turns out to be opposite to the direction of the ξ axis. In so doing, as the potential of isotherms increases, the length of their path along the ξ coordinate decreases, and at $\Theta = 25.99$ the isotherms degenerate into a point positioned on the mark of the coordinate $\xi = 1$ (the stationary regime of heat exchange, which practically takes place already at Fo = 2.0, ensues (see Fig. 11b).

Analysis of the velocity distribution of isotherms permits concluding that for values $0 < \Theta < 1.0$ with approach to the perturbation temperature front the velocities of all isotherms tend to infinite values. This fact is explained by the fulfillment on the line of the perturbation temperature front of the adiabatic wall condition. For $\Theta > 1.0$, because of the appearance of local symmetry of the temperature field, the character of the isotherm distribution changes qualitatively. In particular, isotherms of equal potential arising on the symmetry line move in opposite directions. In so doing, the initial velocities of isotherms tend to infinite values. Then the velocities of isotherms moving in the direction opposite to the ξ axis decrease approaching the zero value at $\xi \to 0$. The velocities of isotherms moving in the

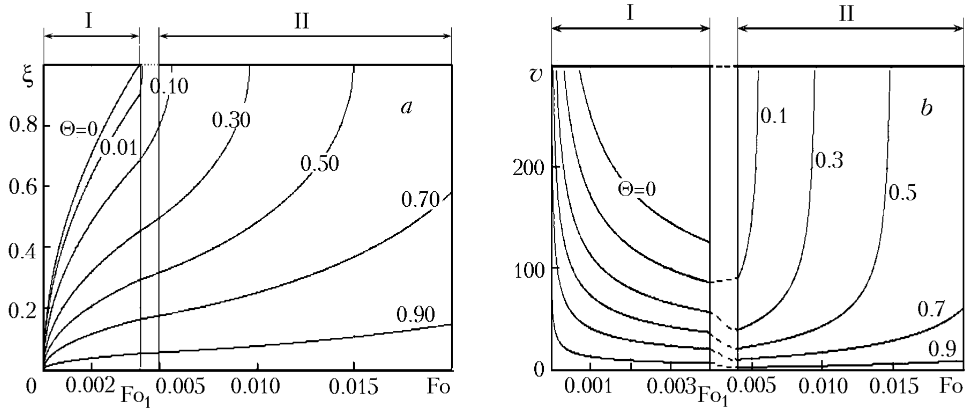

Fig. 13. Distribution of isotherms (a) and velocities of their motion (b) for a nonlinear problem ($\beta = 0.1$). I, II, stages of the process.

direction of the ξ axis, from the infinitely large initial value, first decrease and then, approaching the point $\xi = 1.0$, tend to infinite values again [13].

For all $\Theta > 3.0$, the direction of motion of isotherms is opposite to the direction of the ξ axis. All isotherms arise thereby on the adiabatic wall ($\xi = 1.0$) and have infinitely large initial velocities. Then the velocities of motion decrease, approaching zero values.

The analytical solution of the nonlinear heat conduction problem for an infinite plate at a linear temperature dependence of the thermal diffusivity $a(T) = a_0(1 + \beta T)$ in the second approximation of the first and second stages of the process has the form

$$\Theta_1 (\xi, \text{Fo}) = \Delta T + B_1 \frac{\xi}{q_1} + B_2 \left(\frac{\xi}{q_1}\right)^2 - B_3 \left(\frac{\xi}{q_1}\right)^3 + B_4 \left(\frac{\xi}{q_1}\right)^4 + B_5 \left(\frac{\xi}{q_1}\right)^5 \quad (0 < \text{Fo} \le \text{Fo}_1) , \tag{57}$$

$$\Theta_2 (\xi, \text{Fo}) = \Delta T - \frac{1}{5} (\Delta T - q_2) (8\xi - 4\xi^3 + \xi^4) \quad (\text{Fo}_1 \le \text{Fo} < \infty) , \tag{58}$$

where

$$B_1 = 2r - \sqrt{B} ; \quad B = 10r\Delta T + 4r^2 ; \quad r = 2 (1 + \beta T_w)/\beta ; \quad B_2 = 2 (2\sqrt{B} - 5\Delta T - 4r) ;$$

$$B_3 = 2 (3\sqrt{B} - 10\Delta T - 6r) ; \quad B_4 = 4\sqrt{B} - 15\Delta T - 8r ; \quad B_5 = 4\Delta T + 2r - \sqrt{B} ;$$

$$q_1 (\text{Fo}) = \sqrt{2K_1 \text{Fo}} ; \quad K_1 = -30 \frac{2 (1 + 2\beta T_w) - \beta\sqrt{K} + \beta^2 T_w (2T_w - \sqrt{K})}{5\Delta T\beta + 2 (1 + 2\beta T_w) - \beta\sqrt{K}} ;$$

$$K = (1 + \beta T_w) (5\Delta T\beta + 4 + 4\beta T_w)/\beta^2 .$$

The results of calculations of the distribution of isotherms and velocities of their motion for the nonlinear problem are given in Fig. 13 (for the case of $T_0 = 0^{\circ}$C; $T_w = 100^{\circ}$C; $\beta = 1.0$). Their analysis makes it possible to conclude that at $\beta = 0.1$ the velocities of motion of isotherms of equal potential are much higher than the velocities of motion of isotherms in the linear problem (see Fig. 4b). For example, the time of advance of the isotherm $\Theta = 0.5$ from the coordinate $\xi = 0$ to $\xi = 1$ is Fo = 0.015, whereas in the linear problem this time is Fo = 0.318.

As applied to all the above solutions, we have made a comparison with the exact values of temperatures, and in their absence — with calculations by the finite-difference sweep methods. Analysis of the results of calculations by different methods makes it possible to draw the conclusion that already in the second approximation of the first stage of the process the discrepancy does not exceed 1%, and in the second stage they practically coincide.

CONCLUSIONS

1. By introducing the perturbation temperature front with the use of additional boundary conditions, we have developed a method for obtaining analytical solutions of nonstationary heat conduction problems that makes it possible to find solutions for some problems with a given degree of accuracy throughout the range of change in the Fourier number. The solutions have a simple form of exponential algebraic polynomials containing no special functions. Obtaining such solutions was possible due to the splitting of the heat conduction process into two interrelated processes by introducing two additional sought functions: $q_1(\mathrm{Fo})$ for the first stage of the process and $q_2(\mathrm{Fo})$ for the second stage. The introduction of the function $q_1(\mathrm{Fo})$ characterizing the motion of the perturbation temperature front along the ξ coordinate with time makes it possible to avoid the fulfilment of the initial condition of the form $\Theta(\xi, 0) = 0$ replacing it by the initial condition $q_1(0) = 0$ fulfilled only at the point $\xi = 0$. Such an approach is physically justified by the fact that at the first stage of the process beyond the perturbation temperature front the boundary-value problem is not defined.

2. The additional sought functions $q_1(\mathrm{Fo})$ and $q_2(\mathrm{Fo})$ are introduced in complete compliance with the physical meaning of the boundary-value problem. Both these functions are determinable values in obtaining solutions by any other methods. The need to single out these functions specially is associated with the possibility of splitting the initial boundary-value problem into two interrelated processes: the inertial (irregular) process and the steady-state one, in which the laws of the temperature change with time are so different that combining them within the framework of a single boundary-value problem leads to great difficulties of obtaining its analytical solution.

3. The physical meaning of using the additional boundary conditions is the possibility of the most accurate (depending on the number of boundary conditions — number of approximations) fulfilment of the initial differential equation inside the domain and at its boundary points. This property is already inherent in their deduction based on the requirement that the differential equation and its derivatives be fulfilled exactly at the boundary points and at the perturbation temperature front. Subjection of the sought solution to the additional boundary conditions leads to an exact fulfilment of the differential equation at these points on the ξ coordinate at which at a given instant of time the perturbation temperature front is situated. Since the domain of variability of the perturbation temperature front covers the entire range of change in the spatial coordinate $0 \le q_1(\mathrm{Fo}) \le 1$, then, consequently, the larger the number of boundary conditions used, the more exact will be the fulfillment of the equation inside the domain.

4. The simplicity of the expressions for analytical solutions makes it possible to investigate the boundary-value problems in the fields of isotherms and determine the velocities of their travel with time, which is difficult to do if the classical exact analytical solutions are used for these purposes. Analysis of the distribution of isotherms and velocities of their motion shows that in the vicinity of the boundary points ($\xi \to 0$ and $\xi \to 1$) the velocities of travel of isotherms tend to infinite values, which is due to the setting of idealized boundary conditions: the boundary condition of the first kind (thermal shock) at $\xi = 0$ and the condition of the absence of heat exchange (adiabatic wall condition) at $\xi = 1$. In real practical cases, these boundary conditions cannot be realized exactly — the degree of approximation to them depends on concrete conditions of the heat exchange.

NOTATION

a, a_0, thermal diffusivity and its initial value; $E = \exp\,[-3(\mathrm{Fo} - \mathrm{Fo}_1)]$; $\mathrm{Bi} = \alpha R/\lambda$, Biot number; $\mathrm{Fo} = a\tau/R^2$, Fourier number; $\mathrm{Ki}(\mathrm{Fo}) = \nu(\mathrm{Fo})R/[\lambda(T_\mathrm{w} - T_0)]$, Ki_0, Kirpichev criterion and its initial value; $\mathrm{Pd} = bR^2/(aT_0)$, Predvoditelev criterion; $\mathrm{Po} = \omega R^2/\ [\lambda(T_\mathrm{w} - T_0)]$, Pomerantsev criterion; R, half of the plate thickness, m; T_0, initial temperature, $^\mathrm{o}$C; T_m, temperature of the medium, $^\mathrm{o}$C; T_w, temperature of the wall, $^\mathrm{o}$C; x, coordinate, m; α, heat transfer coefficient, $\mathrm{W/(m^2{\cdot}K)}$; β, coefficient; $\Delta T = T_\mathrm{w} - T_0$, $^\mathrm{o}$C; λ, heat conductivity coefficient, W/(m·K); ν, heat flow, $\mathrm{W/m^2}$; $\xi = x/R$, dimensionless coordinate; τ, time, sec; ω, power of the internal heat source, $\mathrm{W/m^3}$. Subscripts: m, medium; w, wall.

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